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Miyata, Hideaki; Sekine, Jun

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Hideaki Miyata† and Jun Sekine‡
† Department of Mathematics, Kyoto University,
Kitashirakawa-Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan.
e-mail: myt1907@yahoo.co.jp

‡ Graduate School of Engineering Science, Osaka University,
1-3, Machikaneyama, Toyonaka, Osaka 560-8531, Japan.
e-mail: sekine@sigmath.es.osaka-u.ac.jp

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Abstract

A *Bayesian* power-utility maximization is considered, where mean-return-rates of risky assets (or, more precisely, the market price of risk) is an unobservable random vector and the Arrow-Pratt's risk-aversion parameter is larger than 1. It is shown that the optimal expected utility grows *hyperbolically* in the long run if we omit the effect of the risk-free interest rate. This provides a sharp contrast to the results of "non-Bayesian" settings: for instance, in the case of constant market price of risk, the optimal expected power-utility grows *exponentially* in the long run.

Keywords: Bayesian CRRA-utility maximization, partial information, long-term growth rate, hyperbolic growth, hyperbolic discount, Kelly portfolio, fractional Kelly portfolio.

1 Introduction

In the present article, we introduce major findings of Hayashi, Miyata and Sekine (2012), where the expected power-utility maximization of terminal wealth

$$U^{(T,\gamma)}(x) := \sup_{\pi} \mathbb{E} u_{(\gamma)}(X_T^{x,\pi}) \tag{1.1}$$

is considered in a continuous-time financial market, consists of one riskless asset and n-risky assets. Here, we use notation for the CRRA-utility function

$$u_{(\gamma)}(x) := \frac{x^{1-\gamma}}{1-\gamma},$$

where Arrow-Pratt's relative risk-aversion parameter is set as

$$\gamma > 1,$$

and we denote by $X_T^{x,\pi}$ the wealth of a self-financing investor at the terminal date $T \in \mathbb{R}_{++}$, where $x \in \mathbb{R}_{++}$ is an initial wealth and $\pi := (\pi_t)_{t\in[0,T]}$ is a dynamic investment policy. In particular, we assume that the so-called market price of risk vector λ is a hidden, unobservable random variable, which means that (1.1) is a partially-observable (or Bayesian) optimization problem (see Section 2 for the detail of the setup). For this problem, we are interested in the long-term growth rate of optimal expected utility, i.e., writing

$$U^{(T,\gamma)}(x) = u_{(\gamma)}(x) \exp\left\{ \int_0^T \partial_t \log U^{(t,\gamma)}(x) dt \right\},$$

we are interested in the asymptotic behaviour of $\partial_t \log U^{(t,\gamma)}(x)$ as $t \gg 1$. We obtain the following *hyperbolic* long-term growth rate,

$$\partial_T \log U^{(T,\gamma)}(x) = (1-\gamma)r - \frac{n}{2}\frac{1}{T} + \epsilon(T) \quad \text{as } T \to \infty, \tag{1.2}$$

where r is the constant risk-free interest rate and $\epsilon(T)$ is a function "smaller" than $1/T$ as $T \to \infty$ (see Proposition 5.2 and Remark 5.2 for the details). It is interesting to see that (1.2) provides a sharp contrast to the result of "non-Bayesian" case: if the market price of risk vector λ is constant, then, we have the exact expression,

$$\partial_T \log U^{(T,\gamma)}(x) = (1-\gamma)\left(r + \frac{1}{2\gamma}|\lambda|^2 \right),$$

(see (3.3)), i.e., this optimal non-Bayesian power-utility grows exponentially with respect to T, and the (norm) of the market price of risk vector affects the growth rate. It is also interesting to see that the right-hand-side of (1.2) has a "universal" value: it is independent of the law of λ (except for the residual term $\epsilon(T)$), and the hyperbolic term depends on the number n of risky-assets (= the dimension of driving Brownian motion) only.

Remark 1.1 ("Endogenous" Hyperbolic Discounting). The above hyperbolic growth of optimal power-utility plays an interesting role in the lifetime consumption maximization problem,

$$U^{(\gamma)}(x) := \sup_{(\pi,c)} \mathbb{E} \int_0^\infty \exp\left\{ -\int_0^t \rho(u)du \right\} u_{(\gamma)}(c_t)dt, \tag{1.3}$$

which is studied in Miyata (2012), [15]. To consider (1.3), a similar market model with partial information is employed and a Bayesian self-financing investor with the wealth process $(X_t^{x,\pi,c})_{t\geq 0}$ is considered, where $x \in \mathbb{R}_{++}$ is an initial wealth, $\pi := (\pi_t)_{t\geq 0}$ is a dynamic investment policy, and $c := (c_t)_{t\geq 0}$ is a dynamic consumption plan, and, for a given discount rate process $(\rho(t))_{t\geq 0}$, the maximization is considered with respect to both π and c. Actually, with the *hyperbolic* discount rate

$$\rho(t) := \delta + \frac{\beta}{1+\alpha t},$$

we can characterize the critical values $(\underline{\delta}, \underline{\beta})$ to ensure the solvability of (1.3), i.e., $|U^{(\gamma)}(x)| < \infty$ holds if and only if one of the following conditions are satisfied,

$$\text{(a) } \delta > \underline{\delta}, \quad \text{or} \quad \text{(b) } \delta = \underline{\delta} \text{ and } \beta > \underline{\beta}$$

(see [15] for details). Among mathematical finance literatures, Zervos (2008), Bjork and Murgoci (2010), Eckland, Mbodji and Pirvu (2011), and so on treat optimal consumption problems on finite horizon with hyperbolic discounting. They all consider the problems with *a priori* hyperbolic discounting rates. It is interesting to see that, in Bayesian setting, contrarily to the above-mentioned studies, hyperbolic discounting is derived as a natural consequence, i.e., the critical and "minimal" discounting rate

$$\underline{\rho}(t) := \underline{\delta} + \frac{\underline{\beta}}{1+\alpha t}$$

contains a hyperbolic term.

The organization of the present article is as follows. In Section 2, we formulate our financial market model with partial information. After introducing the setup, in Section 3, we mention about the standard baseline results: Merton's optimal power-utility result, which grows exponentially with respect to the terminal time T. In Section 4, we introduce the results on Bayesian CRRA-utility maximization of terminal wealth, which is studied in Karatzas and Zhao (2001). In Section 5, we analyze the long-time asymptotics of the optimal Bayesian CRRA-utility, and observe its hyperbolic growth in power-utility case. In Appendix, we mention about a dynamic programming approach to solve our Bayesian CRRA-utility maximization.

2 Market Model

Consider a continuous-time financial market, consisting of one riskless asset and n-risky assets. The price process $S^0 := (S^0_t)_{t\geq 0}$ of the riskless asset is given by

$$S^0_t := e^{rt}, \tag{2.1}$$

where $r \in \mathbb{R}_{\geq 0}$ is the constant risk-free interest rate. The price process $S := (S^1, \dots, S^n)^\top$, $S^i := (S^i_t)_{t\geq 0}$ of n-risky assets, where $(\cdot)^\top$ denotes the transpose of a vector or a matrix, is defined in the following way: Let $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ be a standard probability space, endowed with the n-dimensional Brownian motion $\tilde{W} := (\tilde{W}^1, \dots, \tilde{W}^n)^\top$, $\tilde{W}^i := (\tilde{W}^i_t)_{t\geq 0}$, where $\tilde{W}_0 \in \mathbb{R}^n$ is constant, and the

n-dimensional random variable λ, which is independent of $\tilde{W}$. The law of λ is denoted by ν, i.e.,

$$\nu(dx) := \tilde{\mathbb{P}}(\lambda \in dx).$$

We alway assume

$$\int_{\mathbb{R}^n} |z|\nu(dz) < \infty \tag{2.2}$$

(or a stronger condition (4.1)). We call this probability space the reference probability space. On $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}, (\mathcal{F}_t)_{t\geq 0})$, where

$$\mathcal{F}_t := \sigma(\tilde{W}_u; u \in [0,t]),$$

we define

$$dS_t = \mathrm{diag}(S_t)\sigma(t, S_t)d\tilde{W}_t, \quad S_0 \in \mathbb{R}^n_{++}, \tag{2.3}$$

where $\sigma : \mathbb{R}_+ \times \mathbb{R}^n_+ \ni (t,y) \mapsto \sigma(t,y) \in \mathbb{R}^{n\times n}$ satisfies $c_1 I \leq \sigma\sigma^\top(t,y) \leq c_2 I$ for any $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ with some constants $0 < c_1 < c_2$ and $\mathrm{diag}(x)$ denotes the diagonal matrix whose (i,i)-element is equal to the i-th element x^i of $x := (x^1, \ldots, x^n)^\top \in \mathbb{R}^n$. The stochastic differential equation (abbreviated to SDE, hereafter) (2.3) has a unique strong solution, which implies that $\mathcal{F}_t \supset \sigma(S_u; u \in [0,t])$. Moreover, we see that

$$\tilde{W}_t = \tilde{W}_0 + \int_0^t \sigma(u, S_u)^{-1} \left\{\mathrm{diag}(S_u)^{-1} dS_u - r\mathbf{1}du\right\}, \tag{2.4}$$

which implies that $\mathcal{F}_t \subset \sigma(S_u; u \in [0,t])$. Hence, we deduce the relation

$$\mathcal{F}_t = \sigma(S_u; u \in [0,t]) \tag{2.5}$$

for all $t \geq 0$. We next define the filtration $(\mathcal{G}_t)_{t\geq 0}$ by

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\lambda), \tag{2.6}$$

and the probability measure $\mathbb{P}$ on $(\Omega, \vee_{t\geq 0}\mathcal{G}_t)$ that satisfies

$$d\mathbb{P}|_{\mathcal{G}_t} = Z_t d\tilde{\mathbb{P}}|_{\mathcal{G}_t} \tag{2.7}$$

for each $t \geq 0$, where

$$Z_t := \exp\left\{\lambda^\top(\tilde{W}_t - \tilde{W}_0) - \frac{|\lambda|^2}{2}t\right\}.$$

We call $\mathbb{P}$ the real-world probability measure. By Cameron-Martin-Maruyama-Girsanov's theorem, the process $W := (W_t)_{t\geq 0}$, given by

$$W_t := \tilde{W}_t - \lambda t, \tag{2.8}$$

is a $(\mathbb{P}, \mathcal{G}_t)$-Brownian motion. Combining (2.3) and (2.8), we obtain the $\mathbb{P}$-dynamics of S,

$$dS_t = \operatorname{diag}(S_t)\left\{\mu(t, S_t)dt + \sigma(t, S_t)dW_t\right\}, \quad S_0 \in \mathbb{R}^n_{++},$$

where we define the mean-return-rate vector

$$\mu(t, y) := r\mathbf{1} + \sigma(t, y)\lambda$$

with $\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^n$. Here, note that W and λ are independent under the real-world probability measure $\mathbb{P}$ since the increment $W_{t_2} - W_{t_1}$ of the $(\mathbb{P}, \mathcal{G}_t)$-Brownian motion W is independent of $\mathcal{G}_0 = \sigma(\lambda)$. Also, note that $\mathbb{P}(\lambda \in A) = \tilde{\mathbb{P}}(\lambda \in A) = \nu(A)$ for any $A \in \mathcal{B}(\mathbb{R}^n)$.

Remark 2.1. For agents having information $(\mathcal{F}_t)_{t\geq 0}$, the market price of risk

$$\lambda = \{\sigma^{-1}(\mu - r\mathbf{1})\}(t, S_t)$$

of S, which is a $\mathcal{G}_0$-measurable random variable, cannot be directly observed. It is a *hidden* variable, which has to be estimated. The law ν of λ is called the prior distribution of λ, and the conditional expectation

$$\hat{\lambda}_t := \mathbb{E}[\lambda|\mathcal{F}_t], \quad t \geq 0, \tag{2.9}$$

where $\mathbb{E}[\cdot]$ denotes expectation with respect to $\mathbb{P}$, is called the Bayesian estimator of λ. From the Bayes rule, we see

$$\hat{\lambda}_t = \frac{\tilde{\mathbb{E}}[Z_t\lambda|\mathcal{F}_t]}{\tilde{\mathbb{E}}[Z_t|\mathcal{F}_t]}, \tag{2.10}$$

where we denote by $\tilde{\mathbb{E}}[\cdot]$, expectation with respect to $\tilde{\mathbb{P}}$. The denominator of (2.10) can be expressed as

$$\tilde{\mathbb{E}}[Z_t|\mathcal{F}_t] = F(t, \tilde{W}_t - \tilde{W}_0), \tag{2.11}$$

where

$$F(t, y) := \int_{\mathbb{R}^n} \exp\left(z^\top y - \frac{|z|^2}{2}t\right)\nu(dz), \tag{2.12}$$

and the numerator of (2.10) is equal to $\nabla F(t, \tilde{W}_t)$. So, we see that

$$\hat{\lambda}_t = \nabla \log F(t, \tilde{W}_t - \tilde{W}_0). \tag{2.13}$$

Remark 2.2. $\tilde{W}$, which is expressed as (2.4), can be interpreted as "cumulative Sharpe ratio of the market":

$$\begin{aligned}\tilde{W}_t =& \tilde{W}_0 + \int_0^t \sigma(s, S_s)^{-1}\left\{\operatorname{diag}(S_s)^{-1}dS_s - r\mathbf{1}dt\right\}\\ =& \tilde{W}_0 + \int_0^t (\text{“vol. matrix”})_s^{-1}(\text{“return of } S_s\text{”} - \text{“return of } S_s^0\text{”}\mathbf{1}).\end{aligned}$$

Next, on this financial market, consider a self-financing investor whose available information flow is $(\mathcal{F}_t)_{t\geq 0}$. The wealth process $X^{x,\pi} := (X_t^{x,\pi})_{t\geq 0}$ of the investor is defined by the SDE

$$(2.14) \qquad dX_t^{x,\pi} = X_t^{x,\pi}\left\{\sum_{i=1}^n \pi_t^i \frac{dS_t^i}{S_t^i} + \left(1-\sum_{i=1}^n \pi_t^i\right)\frac{dS_t^0}{S_t^0}\right\}, \quad X_0^{x,\pi} = x,$$

where $x \in \mathbb{R}_{++}$ is the initial wealth of the investor and $\pi := (\pi^1, \dots, \pi^n)^\top$, $\pi^i := (\pi_t^i)_{t\geq 0}$ is a dynamic investment strategy of the investor, which is $\mathcal{F}_t$-adapted. For a given finite time horizon $T \in \mathbb{R}_{++}$ and initial wealth $x \in \mathbb{R}_{++}$, consider the utility maximization of terminal wealth,

$$(2.15) \qquad U^{(T,\gamma)}(x) := \sup_{\pi \in \mathscr{A}_T} \mathbb{E}u_{(\gamma)}\left(X_T^{x,\pi}\right),$$

where

$$(2.16) \qquad u_{(\gamma)}(x) := \begin{cases} \dfrac{x^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0, \neq 1, \\ \log x & \text{if } \gamma = 1 \end{cases}$$

is the CRRA-utility function with Arrow-Pratt's relative risk-aversion parameter γ and

$$\mathscr{A}_T := \left\{ (f_t)_{t\in[0,T]} \;\middle|\; \begin{array}{l} n\text{-dimensional } \mathcal{F}_t\text{-progressively measurable,} \\ \int_0^T |f_t|^2 dt < \infty \text{ a.s.} \end{array} \right\}$$

is the totality of admissible investment strategies.

Remark 2.3. Similar market models with partial information, where unobservable random market price of risks are employed, are studied in Brennan and Xia (2001), Cvitanić et. al. (2006), Karatzas (1997), Karatzas and Zhao (2001), Lakner (1995), Pham and Quenez (2001), Rieder and Bäuerle (2005), Xia (2001), and Zohar (2001), for example.

3 Exponential Growth of Optimal Power-utility with Constant λ

Before analyzing Bayesian utility maximization (2.15), in this section, we consider a special "non-Bayesian" situation: let $\nu(dx) := \delta_{\lambda_0}(dx)$ be the Dirac's delta measure, i.e., let $\lambda :\equiv \lambda_0 \in \mathbb{R}^n$ be a constant vector. Then, the solution to (2.15), which has been originally investigated by Merton (1969, 1971), is now well-known and described as follows:

(A) The optimal investment strategy $\hat{\pi}^{(\gamma)} := (\hat{\pi}_t^{(\gamma)})_{t\in[0,T]} \in \mathscr{A}_T$ is given by

$$(3.1) \qquad \hat{\pi}_t^{(\gamma)} := \frac{1}{\gamma}(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})(t, S_t) = \frac{1}{\gamma}(\sigma^\top)^{-1}(t, S_t)\lambda.$$

(B) The associated optimal wealth process $(\hat{X}_t^{(\gamma)})_{t\in[0,T]}$ is written as

$$(3.2)\qquad \hat{X}_t^{(\gamma)} := X_t^{x,\hat{\pi}^{(\gamma)}} = x\exp\left\{\frac{1}{\gamma}\lambda^\top (W_t+\lambda t) + \left(r - \frac{1}{2\gamma^2}|\lambda|^2\right)t\right\}.$$

(C) The optimal utility is computed as

$$(3.3)\qquad U^{(T,\gamma)}(x) = \mathbb{E}u_{(\gamma)}(\hat{X}_T^{(\gamma)}) = u_{(\gamma)}\left(xe^{(r+\frac{1}{2\gamma}|\lambda|^2)T}\right).$$

In this constant λ case, since the optimal strategy (3.1) is T-independent, we can re-define the investment strategy $\hat{\pi}^{(\gamma)} := (\hat{\pi}_t^{(\gamma)})_{t\geq 0}$ by (3.1) and the wealth process $\hat{X}^{(\gamma)} := (\hat{X}_t^{(\gamma)})_{t\geq 0}$ by (3.2) on the time interval $[0,\infty)$. Let

$$\mathscr{A} := \left\{(f_t)_{t\geq 0};\ (f_t)_{t\in[0,T]} \in \mathscr{A}_T \text{ for all } T>0\right\}.$$

We then see the following long-term optimalities of $\hat{X}^{(\gamma)}$. We note that, for obtaining (II)–(IV) below, the *exponential growth* of optimal power-utility (3.3) (with $\gamma\neq 1$) with respect to T is essential.

(I) (*Maximizing long-term growth rate*). We have that, for any $\pi\in\mathscr{A}$,

$$\varlimsup_{T\to\infty}\frac{1}{T}\log X_T^{x,\pi} \leq \lim_{T\to\infty}\frac{1}{T}\log\hat{X}_T^{(1)} = -\Gamma'(1)\quad \text{a.s..}$$

where

$$-\Gamma'(1) := r + \frac{1}{2}|\lambda|^2.$$

For the proof, see Theorem 3.10.1 of Karatzas and Shreve (1998) [8].

(II) (*Maximizing long-term growth rate of expected power-utility*). When $0<\gamma<1$, we see that, for any $\pi\in\mathscr{A}$,

$$\varlimsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}u_{(\gamma)}(X_T^{x,\pi}) \leq \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}u_{(\gamma)}(\hat{X}_T^{(\gamma)}) = \Gamma(\gamma).$$

When $\gamma>1$, we see that, for any $\pi\in\mathscr{A}$,

$$\varliminf_{T\to\infty}\frac{1}{T}\log\mathbb{E}u_{(\gamma)}(X_T^{x,\pi}) \geq \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}u_{(\gamma)}(\hat{X}_T^{(\gamma)}) = \Gamma(\gamma).$$

Here, we set

$$\Gamma(\gamma) := (1-\gamma)\left(r + \frac{1}{2\gamma}|\lambda|^2\right).$$

(III) (*Maximizing long-term upside-chance large deviation probability*). Let $0<\gamma<1$. We have that, for any $\pi\in\mathscr{A}$,

$$\varlimsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left(\frac{1}{T}\log X_T^{x,\pi}\geq k(\gamma)\right) \leq \lim_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left(\frac{1}{T}\log\hat{X}_T^{(\gamma)}\geq k(\gamma)\right),$$

where the target growth rate $k(\gamma) > -\Gamma'(1)$ is defined by

$$k(\gamma) := -\Gamma'(\gamma) = r + \frac{1}{2\gamma^2}|\lambda|^2. \tag{3.4}$$

For the details, see Pham (2003).

(IV) (*Minimizing long-term downside-risk large deviation probability*). Let $\gamma > 1$. We have that, for any $\pi \in \mathscr{A}$,

$$\varliminf_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\frac{1}{T}\log X_T^{x,\pi} \le k(\gamma)\right) \ge \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\frac{1}{T}\log \hat{X}_T^{(\gamma)} \le k(\gamma)\right),$$

where the target growth rate $r < k(\gamma) < -\Gamma'(1)$ is defined by (3.4). For the details, see Hata et. al. (2010).

Remark 3.1 (Kelly and fractional Kelly portfolios). The log-optimal portfolio $\hat{\pi}^{(1)}$ is sometimes called the GOP (Growth Optimal Portfolio) by the property (I), or the Kelly portfolio. The latter name comes from the pioneer work by Kelly (1956) [10]: the optimality in (I) can be interpreted as a corollary of the result obtained in [10]. Also, the power-optimal portfolio $\hat{\pi}^{(\gamma)}$ with the risk-aversion parameter $\gamma > 1$ is sometimes called the fractional Kelly portfolio, which has been proposed to decrease the "risky" features of the "full" Kelly portfolio $\hat{\pi}^{(1)}$: see Chapter IV and Section 27 of Chapter III of Maclean et. al. (2011) [12]. Note that the above (II) and (IV) characterize long-term optimalities of the fractional Kelly portfolio $\hat{\pi}^{(\gamma)}$ $(\gamma > 1)$.

4 Bayesian CRRA Utility Maximization

In this section, we introduce the solution to our Bayesian CRRA-utility maximization (2.15), which has been obtained in Karatzas and Zhao (2001), [9], in an essential form.

For $\gamma = 1$, i.e., log-utility case, we see the following.

Theorem 4.1 (Theorem 3.2, Example 3.3 and 4.4 of [9]). *Assume*

$$\int_{\mathbb{R}^n} |z|^2 \nu(dz) < \infty. \tag{4.1}$$

For any $T, x \in \mathbb{R}_{>0}$, the following are valid.
1. *The optimal wealth process $\hat{X}^{(T,1)} := (\hat{X}_t^{(T,1)})_{t\in[0,T]}$ is given by*

$$\hat{X}_t^{(T,1)} = x e^{rt} F(t, \tilde{W}_t - \tilde{W}_0), \tag{4.2}$$

where we use (2.12).
2. *The optimal strategy $\hat{\pi}^{(T,1)} := (\hat{\pi}_t^{(T,1)})_{t\in[0,T]}$ that satisfies*

$$\hat{X}_t^{(T,1)} = X_t^{x,\hat{\pi}^{(T,1)}} \quad t \in [0,T]$$

is given by

$$\hat{\pi}_t^{(T,1)} := (\sigma(t,S_t)^\top)^{-1}\left(\frac{\nabla F}{F}\right)(t,\tilde{W}_t-\tilde{W}_0) = (\sigma(t,S_t)^\top)^{-1}\hat{\lambda}_t, \tag{4.3}$$

where we use (2.9).

3. *The optimal expected utility is expressed as*

$$\begin{aligned} U^{(T,1)}(x) &= \log x + rT + \tilde{\mathbb{E}}F(T,\tilde{W}_T-\tilde{W}_0)\log F(T,\tilde{W}_T-\tilde{W}_0) \\ &= \log x + rT + \frac{1}{2}\mathbb{E}\int_0^T |\hat{\lambda}_t|^2 dt. \end{aligned}$$

Proof. The first two assertions are derived directly from Example 3.3 of [9]. To see the third assertion, we first deduce

$$\tilde{\mathbb{E}}F(T,\tilde{W}_T-\tilde{W}_0)\log F(T,\tilde{W}_T-\tilde{W}_0) = \mathbb{E}\log F(T,\tilde{W}_T-\tilde{W}_0).$$

Use Itô's formula and (2.13) to see that

$$\begin{aligned} \log F(t,\tilde{W}_t-\tilde{W}_0) &= \int_0^t \hat{\lambda}_u^\top d\tilde{W}_u - \frac{1}{2}\int_0^t |\hat{\lambda}_u|^2 du \\ &= \int_0^t \hat{\lambda}_u^\top dB_u + \frac{1}{2}\int_0^t |\hat{\lambda}_u|^2 du, \end{aligned} \tag{4.4}$$

where

$$B_t := \tilde{W}_t - \int_0^t \hat{\lambda}_u du, \quad t \geq 0$$

is a $(\mathbb{P},\mathcal{F}_t)$-Brownian motion by Cameron-Martin-Maruyama-Girsanov's formula. Note that

$$\mathbb{E}\int_0^t |\hat{\lambda}_u|^2 du = \int_0^t du \mathbb{E}\left|\mathbb{E}[\lambda|\mathcal{F}_u]\right|^2 \leq t\mathbb{E}|\lambda|^2 < \infty$$

for any $t > 0$. Hence, it follows that

$$\mathbb{E}\log F(t,\tilde{W}_t-\tilde{W}_0) = \frac{1}{2}\mathbb{E}\int_0^T |\hat{\lambda}_t|^2 dt.$$

□

Next, we consider a power-utility, which is more risk-averse than the log-utility, i.e., employ (2.16) with

$$\gamma \in (1,\infty). \tag{4.5}$$

To treat this situation, we introduce, for $0 \leq t \leq T < \infty$,

$$\begin{aligned} G^{(T,\gamma)}(t,y) :=& \tilde{\mathbb{E}}\left[F(T,y+\tilde{W}_t-\tilde{W}_0)^{\frac{1}{\gamma}}\right] \\ =& \int_{\mathbb{R}^n} F(T,y+\sqrt{t}z)^{\frac{1}{\gamma}}\frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{|z|^2}{2}}dz \end{aligned} \tag{4.6}$$

where we use (2.12). Here, recalling that

$$F(t,y) = \exp\left(\frac{1}{2t}|y|^2\right)\int_{\mathbb{R}^n} \exp\left(-\frac{t}{2}\left|z-\frac{y}{t}\right|^2\right)\nu(dz) \le \exp\left(\frac{1}{2t}|y|^2\right),$$

we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} F(T, y+\sqrt{t}z)^{\frac{1}{\gamma}} \frac{1}{(2\pi)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|z|^2}{2}} dz \\
\le &\int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left\{\frac{|y+\sqrt{t}z|^2}{2\gamma T} - \frac{|z|^2}{2}\right\} dz \\
= &\left(\frac{\gamma T}{\gamma T - t}\right)^{\frac{n}{2}} \exp\left\{\frac{|y|^2}{2(\gamma T - t)}\right\},
\end{aligned}$$

hence, the integral in (4.6) has a finite value. Moreover, we see that, for $0 \le t \le T < \infty$,

$$\begin{aligned}
\nabla G^{(T,\gamma)}(t,y) =&\tilde{\mathbb{E}}\left[(\nabla F \cdot F^{\frac{1-\gamma}{\gamma}})(T, y + \tilde{W}_t - \tilde{W}_0)\right] \\
=&\int_{\mathbb{R}^n} (\nabla F \cdot F^{\frac{1-\gamma}{\gamma}})(T, y+\sqrt{t}z) \frac{1}{(2\pi)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|z|^2}{2}} dz
\end{aligned}$$

and that the above integral has a finite value. Indeed, when $t = 0$, these equalities are trivial, and, for $0 < t \le T$, we deduce that

$$|\nabla F(t,y)| \le \exp\left(\frac{1}{2t}|y|^2\right)\int_{\mathbb{R}^n} |z| \exp\left(-\frac{t}{2}\left|z-\frac{y}{t}\right|^2\right)\nu(dz) \le \exp\left(\frac{1}{2t}|y|^2\right)\tilde{\mathbb{E}}|\lambda|$$

and that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|(\nabla F \cdot F^{\frac{1-\gamma}{\gamma}})(T, y+\sqrt{t}z)\right| \frac{1}{(2\pi)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|z|^2}{2}} dz \\
\le&\tilde{\mathbb{E}}|\lambda| \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left\{\frac{|y+\sqrt{t}z|^2}{2\gamma T} - \frac{|z|^2}{2}\right\} dz \\
=&\tilde{\mathbb{E}}|\lambda| \left(\frac{\gamma T}{\gamma T - t}\right)^{\frac{n}{2}} \exp\left\{\frac{|y|^2}{2(\gamma T - t)}\right\}.
\end{aligned}$$

We now see the following.

Theorem 4.2 (Theorem 3.2, Example 3.5 and 4.6 of [9]). *Assume (2.2) and (4.5). For any $T, x \in \mathbb{R}_{>0}$, the following are valid.*
1. *The optimal wealth process $\hat{X}^{(T,\gamma)} := (\hat{X}_t^{(T,\gamma)})_{t\in[0,T]}$ is given by*

$$\hat{X}_t^{(T,\gamma)} = x\mathrm{e}^{rt}\frac{G^{(T,\gamma)}(T-t, \tilde{W}_t - \tilde{W}_0)}{G^{(T,\gamma)}(T,0)}, \tag{4.7}$$

where we use (4.6).

2. *The optimal strategy* $\hat{\pi}^{(T,\gamma)} := (\hat{\pi}_t^{(T,\gamma)})_{t\in[0,T]}$ *that satisfies*

$$\hat{X}_t^{(T,\gamma)} = X_t^{x,\hat{\pi}^{(T,\gamma)}} \quad t \in [0,T]$$

is given by

$$\hat{\pi}_t^{(T,\gamma)} := (\sigma(t,S_t)^\top)^{-1}\frac{\nabla G^{(T,\gamma)}(T-t,\tilde{W}_t-\tilde{W}_0)}{G^{(T,\gamma)}(T-t,\tilde{W}_t-\tilde{W}_0)}. \tag{4.8}$$

3. *The optimal expected utility is expressed as*

$$U^{(T,\gamma)}(x) = u_{(\gamma)}(xe^{rT})\left\{G^{(T,\gamma)}(T,0)\right\}^\gamma. \tag{4.9}$$

In [9], the so-called martingale method is employed to solve this problem with partial information. In Appendix, we describe a different solution method, using a standard dynamic programming.

Example 4.1 (Gaussian Prior). This example treats a slight generalization of computations demonstrated in Cvitanić et. al. (2006). Let $\nu \sim N(l,L)$, i.e.,

$$\nu(dz) := \frac{1}{(2\pi)^{n/2}\sqrt{\det(L)}}\exp\left\{-\frac{1}{2}(z-l)^\top L^{-1}(z-l)\right\}dz,$$

where $L \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite covariance matrix and $l \in \mathbb{R}^n$ is a mean vector. Then, the Bayesian estimator (2.9), which is expressed as (2.13), is computed as

$$\hat{\lambda}_t = \nabla \log F(t,\tilde{W}_t). = (L^{-1}+tI)^{-1}(\tilde{W}_t + L^{-1}l). \tag{4.10}$$

Indeed, we see $F(t,y)$, given by (2.12), is computed as

$$\begin{aligned} F(t,y) =& \exp\left\{\frac{1}{2}(y+L^{-1}l)^\top(L^{-1}+tI)^{-1}(y+L^{-1}l)\right. \\ &\left. -\frac{1}{2}l^\top L^{-1}l - \frac{1}{2}\log\det(I+tL)\right\}. \end{aligned}$$

Let

$$K_t := (L^{-1}+tI)^{-1} = L(I+tL)^{-1} = (I+tL)^{-1}L$$

and define, for $\theta \in (0,1)$,

$$P_t^{(T)}(\theta) = K_T\left\{t\theta(I-t\theta K_T)^{-1} + K_T^{-1}\right\}K_T.$$

We recall that

$$\begin{aligned} I - t\theta K_T =& I - t\theta(I+TL)^{-1}L \\ =& (I+TL)^{-1}\{I+(T-t\theta)L\} \\ \geq& (I+TL)^{-1} > 0 \end{aligned}$$

for all $t \in [0,T]$ and $\theta \in (0,1)$. So, $P_t^{(T)}(\theta)$ is a well-defined, symmetric and positive definite matrix. Also, we may notice that $(P_t^{(T)}(\theta))_{t\in[0,T]}$ solves the differential Riccati equation

$$\frac{d}{dt}P = \theta P^2, \quad P_0 = K_T.$$

Using these functions, we deduce that (4.6) is calculated as

$$\begin{aligned} G^{(T,\gamma)}&(t,y) \\ =\exp&\left[\frac{\theta}{2}(L^{-1}l+y)^\top P_t^{(T)}(\theta)(L^{-1}l+y)\right. \\ &\left.-\frac{\theta}{2}\left\{l^\top L^{-1}l+\log\det(I+TL)\right\}-\frac{1}{2}\log\det\left(I-t\theta K_T\right)\right], \end{aligned} \tag{4.11}$$

where we set

$$\theta := \frac{1}{\gamma}.$$

Indeed, defining

$$\begin{aligned} k_t :=& K_t L^{-1}l \quad \text{and} \\ \kappa_t :=& l^\top L^{-1}K_tL^{-1}l - l^\top L^{-1}l - \log\det(I+tL), \end{aligned}$$

we see

$$\begin{aligned} &G^{(T,\gamma)}(t,y) \\ =&\exp\left\{\theta\left(\frac{1}{2}y^\top K_T y + k_T^\top y + \frac{1}{2}\kappa_T\right)\right\} \\ &\times\int_{\mathbb{R}^n}\frac{1}{(2\pi)^{\frac{n}{2}}}\exp\left\{\theta\left(\frac{t}{2}z^\top K_T z + \sqrt{t}(k_T+K_T y)^\top z\right) - \frac{1}{2}|z|^2\right\}dz \\ =&\exp\left[\theta\left(\frac{1}{2}y^\top K_T y + k_T^\top y + \frac{1}{2}\kappa_T\right) - \frac{1}{2}\log\det\left(I-t\theta K_T\right)\right. \\ &\left.+\frac{t\theta^2}{2}(L^{-1}l+y)^\top K_T\left(I-t\theta K_T\right)^{-1}K_T(L^{-1}l+y)\right], \end{aligned}$$

hence, the expression (4.11) is obtained from this calculation. Inserting (4.11) into (4.8) in Theorem 4.2 and combining this with (4.10), we obtain

$$\begin{aligned} \hat{\pi}_t^{(T,\gamma)} =& \frac{1}{\gamma}(\sigma^\top)^{-1}(t,S_t)P_{T-t}^{(T)}\left(\tfrac{1}{\gamma}\right)(\tilde{W}_t - \tilde{W}_0 + L^{-1}l) \\ =& \frac{1}{\gamma}(\sigma^\top)^{-1}(t,S_t)P_{T-t}^{(T)}\left(\tfrac{1}{\gamma}\right)(L^{-1}+tI)\hat{\lambda}_t. \end{aligned}$$

5 Long-time Asymptotics

5.1 Log-optimal Case

First, consider the log-utility case, assuming (4.1). Then, both the optimal wealth (4.2) and the optimal investment strategy (4.3) are T-independent. So, we re-define

$$\hat{X}^{(1)} := (\hat{X}_t^{(1)})_{t\geq 0}$$

by (4.2) and

$$\hat{\pi}^{(1)} := (\hat{\pi}_t^{(1)})_{t\geq 0}$$

by (4.3). We may assume that $\mathcal{F}_t$ $(t \geq 0)$ is the $\mathbb{P}$-completion of $\sigma(\tilde{W}_u; u \leq t)$. We then see the following.

Proposition 5.1 (Example 5.1 of [9]). *For any $\pi \in \mathscr{A}$, it holds that*

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\log X_T^{x,\pi} \leq \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\log \hat{X}_T^{(1)} = r + \frac{1}{2}\mathbb{E}|\lambda|^2. \tag{5.1}$$

Proof. From Proposition 4.1, we can deduce that

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\log X_T^{x,\pi} \leq \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\log \hat{X}_T^{(1)}$$

holds for any $\pi \in \mathscr{A}$. Using (4.4) and Fubini's theorem, we can see that the right-hand-side of the above is equal to

$$r + \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_0^T \mathbb{E}|\hat{\lambda}_t|^2 dt. \tag{5.2}$$

Recalling the definition (2.9) of $\hat{\lambda}_t$, we can apply the martingale convergence theorem to $(\hat{\lambda}_t)_{t\geq 0}$ to deduce that

$$\hat{\lambda}_\infty := \lim_{t\to\infty} \hat{\lambda}_t = \mathbb{E}[\lambda|\mathcal{F}_\infty] \quad \mathbb{P}\text{-a.s.}, \tag{5.3}$$

where $\mathcal{F}_\infty := \vee_{t\geq 0}\mathcal{F}_t$. Moreover, we see that $\hat{\lambda}_\infty = \lambda$, $\mathbb{P}$-a.s.. Actually, from (2.8), we see that

$$\frac{1}{t}\tilde{W}_t = \frac{1}{t}W_t + \lambda.$$

From this, we deduce that $\lim_{t\to\infty} \frac{1}{t}\tilde{W}_t = \lambda$, $\mathbb{P}$-a.s. since we have $\lim_{t\to\infty} \frac{1}{t}W_t = 0$, $\mathbb{P}$-a.s. by the strong law of large numbers. Hence, the $\mathscr{F}_\infty$-measurability of λ follows. So, we deduce that (5.2) is equal to the right-hand-side of (5.1). $\square$

Remark 5.1. We can also deduce that, for any $\pi \in \mathscr{A}$,

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\log X_T^{x,\pi} \leq \overline{\lim_{T\to\infty}} \frac{1}{T}\log \hat{X}_T^{(1)} = r + \frac{1}{2}|\lambda|^2 \quad \mathbb{P}\text{-a.s.}.$$

Indeed, the above inequality follows from Theorem 3.1 of [8]. To derive the expression of the right-hand-side of the above, using (4.4), we write as

$$\frac{1}{T}\log \hat{X}_T^{(1)} = r + \frac{1}{T}M_T + \frac{1}{2T}\langle M\rangle_T,$$

where we define

$$M_t := \int_0^t \hat{\lambda}_u^\top dB_u.$$

We first deduce that

$$\lim_{T\to\infty}\frac{1}{T}\langle M\rangle_T = \lim_{T\to\infty}\frac{1}{T}\int_0^T |\hat{\lambda}_t|^2 dt = |\lambda|^2 \quad \mathbb{P}\text{-a.s.},$$

where we recall $\hat{\lambda}_t \to \lambda$, $\mathbb{P}$-a.s. as $t\to\infty$. We next deduce that

$$\lim_{T\to\infty}\frac{1}{T}M_T = \lim_{T\to\infty}\frac{M_T}{\langle M\rangle_T}\frac{\langle M\rangle_T}{T} = 0,$$

where the strong law of large numbers for square-integrable martingales is applied, recalling $\lim_{T\to\infty}\langle M\rangle_T = \infty$ on $\{\lambda \neq 0\}$.

Remark 5.2. The portfolio $\hat{\pi}^{(1)} \in \mathscr{A}$ is sometimes called the Bayesian Kelly portfolio.

5.2 Power-optimal Case

We next consider power-utility case. In this subsection, in addition to (2.2) and (4.5), we assume that

(5.4) $\quad \nu(dz) = f_\nu(z)dz$ with $f_\nu \in L^\infty(\mathbb{R}^n)$, which is continuous at $0 \in \mathbb{R}^n$.

We define two functions,

$$\phi(x) := \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{|x|^2}{2}\right),$$
$$\psi(t,x) := \left(\frac{t}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left(-\frac{t}{2}|x-z|^2\right)\nu(dz),$$

recalling that

$$\tilde{\mathbb{E}}h\left(\frac{\tilde{W}_t}{t}+\lambda\right) = \int_{\mathbb{R}^n} h(x)\psi(t,x)dx$$

for any bounded Borel measurable function $h : \mathbb{R} \to \mathbb{R}$. Using these, we see that

$$
\begin{aligned}
(5.5)\quad & G^{(T,\gamma)}(t,y) \\
&= \int_{\mathbb{R}^n} \exp\left(\frac{|y+\sqrt{t}z|^2}{2\gamma T}\right) \left\{ \left(\frac{2\pi}{T}\right)^{\frac{n}{2}} \psi\left(T, \frac{y+\sqrt{t}z}{T}\right) \right\}^{\frac{1}{\gamma}} \phi(z)dz \\
&= \left(\frac{2\pi}{T}\right)^{\frac{n}{2\gamma}} \left(\frac{\gamma T}{\gamma T - t}\right)^{\frac{n}{2}} \exp\left\{\frac{|y|^2}{2(\gamma T - t)}\right\} \\
&\times \int_{\mathbb{R}^n} \left(\frac{\gamma T - t}{2\pi\gamma T}\right)^{\frac{n}{2}} \exp\left(-\frac{\gamma T - t}{2\gamma T}\left|z - \frac{\sqrt{t}}{\gamma T - t} y\right|^2\right) \psi\left(T, \frac{y+\sqrt{t}z}{T}\right)^{\frac{1}{\gamma}} dz.
\end{aligned}
$$

Also, when $t = T$, we see that

$$
(5.6)\qquad G^{(T,\gamma)}(T,y) = \left(\frac{2\pi}{T}\right)^{\frac{n}{2\gamma}} \left(\frac{\gamma}{\gamma-1}\right)^{\frac{n}{2}} \exp\left\{\frac{|y|^2}{2(\gamma-1)T}\right\} \Psi^{(\gamma)}(T,y),
$$

where we define

$$
\begin{aligned}
&\Psi^{(\gamma)}(T,y) := \\
&\int_{\mathbb{R}^n} \left(\frac{\gamma-1}{2\pi\gamma T}\right)^{\frac{n}{2}} \exp\left(-\frac{\gamma-1}{2\gamma T}\left|z - \frac{1}{\gamma-1} y\right|^2\right) \psi\left(T, \frac{y+z}{T}\right)^{\frac{1}{\gamma}} dz.
\end{aligned}
$$

We deduce the following.

Lemma 5.1. (1) $|\psi(t,y)| \le \|f_\nu\|_\infty$ *and* $|\Psi^{(\gamma)}(T,y)| \le \|f_\nu\|_\infty^{\frac{1}{\gamma}}$ *for* $(T,y) \in \mathbb{R}_+ \times \mathbb{R}^n$.
(2) $\lim_{t\to\infty} \psi(t,x) = f_\nu(x)$ *for each* $x \in \mathbb{R}^n$, *if* f_ν *is continuous at* $x \in \mathbb{R}^n$.
(3) $\lim_{T\to\infty} \Psi^{(\gamma)}(T,y) = f_\nu(0)^{\frac{1}{\gamma}}$ *for each* $y \in \mathbb{R}^n$.

Proof. (1) The assertion is straightforward to see.
(2) We see that

$$
\lim_{t\to\infty} \psi(t,x) = \lim_{t\to\infty} \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}|y|^2\right) f_\nu\left(x - \frac{1}{\sqrt{t}} y\right) dy = f_\nu(x)
$$

by the dominated convergence theorem.
(3) We see that

$$
\begin{aligned}
&\left|\Psi^{(\gamma)}(T,y) - f_\nu(0)^{\frac{1}{\gamma}}\right| \\
&\le \int_{\mathbb{R}^n} \left(\frac{\gamma-1}{2\pi\gamma T}\right)^{\frac{n}{2}} \exp\left(-\frac{\gamma-1}{2\gamma T}\left|z - \frac{1}{\gamma-1} y\right|^2\right) \left|\psi\left(T, \frac{y+z}{T}\right)^{\frac{1}{\gamma}} - f_\nu(0)^{\frac{1}{\gamma}}\right| dz.
\end{aligned}
$$

By the dominated convergence theorem, the desired assertion follows. □

With the help of this lemma, we obtain the following. Recall a notation in asymptotic analysis: we write "$a(T) \sim b(T)$ as $T \to \infty$" when $\lim_{T\to\infty} a(T)/b(T) = 1$ holds.

Proposition 5.2. *It holds that*

$$
\begin{aligned}
(5.7) \qquad U^{(T,\gamma)}(x) &= u_{(\gamma)}(xe^{rT}) \left\{ \frac{2\pi}{T} \left(\frac{\gamma}{\gamma-1} \right)^{\gamma} \right\}^{\frac{n}{2}} \Psi^{(\gamma)}(T,0)^{\gamma} \\
&\sim u_{(\gamma)}(xe^{rT}) \left\{ \frac{2\pi}{T} \left(\frac{\gamma}{\gamma-1} \right)^{\gamma} \right\}^{\frac{n}{2}} f_{\nu}(0) \quad \text{as } T \to \infty
\end{aligned}
$$

and that

$$
\begin{aligned}
(5.8) \qquad \hat{X}_T^{(T,\gamma)} &= x \exp\left\{ \frac{n}{2} \log\left(1 - \frac{1}{\gamma}\right) + rT - \log \Psi^{(\gamma)}(T,0) \right. \\
&\quad \left. + \frac{1}{2\gamma T} |\tilde{W}_T - \tilde{W}_0|^2 + \frac{1}{\gamma} \log \psi\left(T, \frac{\tilde{W}_T - \tilde{W}_0}{T}\right) \right\}.
\end{aligned}
$$

Proof. (5.7) is derived from (4.9) and (5.6), using Lemma 5.1. (5.8) is computed from (4.7), (5.5), and (5.6). □

Remark 5.3 (Hyperbolic Growth). From (5.7), we see that

$$
\partial_T \log U^{(T,\gamma)}(x) = (1-\gamma)r - \frac{n}{2}\frac{1}{T} + \epsilon(T),
$$

where we set $\epsilon(T) := \gamma \partial_T \log \Psi^{(\gamma)}(T,0)$. The residual term $\epsilon(T)$ is "smaller" than $1/T$ as $T \to \infty$ in the sense that

$$
\lim_{T\to\infty} \left| \int_1^T \epsilon(t)dt \right| = \left| \log f_{\nu}(0) - \log \Psi^{(\gamma)}(1,0) \right| < \infty,
$$

which is deduced from Lemma 5.1 (3).

Remark 5.4 (Bayesian Fractional Kelly Portfolio). Let $\gamma > 1$. Define the Bayesian fractional Kelly portfolio $\tilde{\pi}^{(\gamma)} := (\tilde{\pi}_t^{(\gamma)})_{t\geq 0} \in \mathscr{A}$ by

$$
\tilde{\pi}_t^{(\gamma)} := \frac{1}{\gamma} (\sigma^{\top})^{-1}(t, S_t) \hat{\lambda}_t.
$$

As mentioned in Remark 3.1, from a practical point of view, this portfolio may be a candidate for long-term "risk-averse" investment. Write the associated wealth process as $\tilde{X}^{(\gamma)} := (\tilde{X}_t^{(\gamma)})_{t\geq 0}$, i.e.,

$$
\tilde{X}_t^{(\gamma)} := X_t^{x,\tilde{\pi}^{(\gamma)}} \quad t \geq 0.
$$

We can deduce that

$$\log \tilde{X}_T^{(\gamma)} = \log \hat{X}_T^{(T,\gamma)} + \log G^{(T,\gamma)}(T,0) + \frac{\gamma-1}{2\gamma^2}\int_0^T |\hat{\lambda}_t|^2 dt$$

and that

$$\lim_{T\to\infty} \frac{1}{T}\left\{\log \tilde{X}_T^{(\gamma)} - \log \hat{X}_T^{(T,\gamma)}\right\} = \frac{\gamma-1}{2\gamma^2}|\lambda|^2 \quad \mathbb{P}\text{-a.s.}.$$

5.3 Cost of Uncertainty

In this subsection, we evaluate a "cost of uncertainty of λ" in the long run, which is proposed and discussed in [9]: We consider an "inside" investor, whose available information flow is $(\mathcal{G}_t)_{t\geq 0}$, where we use (2.6). Note that, for this insider, the market price of risk vector λ is observable at time 0 and so, it can be regarded as a constant. The insider's CRRA-utility maximization is described as

$$\sup_{\pi\in\mathscr{A}_T^{\mathcal{G}}} \mathbb{E}\left[u_{(\gamma)}(X_T^{x,\pi})|\mathcal{G}_0\right],$$

where maximization is considered over the space $\mathscr{A}_T^{\mathcal{G}}$ of n-dimensional $\mathcal{G}_t$-progressively measurable processes $(f_t)_{t\in[0,T]}$ so that $\int_0^T |f_t|^2 dt < \infty$ a.s. From (3.3), we deduce that

$$\sup_{\pi\in\mathscr{A}_T^{\mathcal{G}}} \mathbb{E}\left[u_{(\gamma)}(X_T^{x,\pi})|\mathcal{G}_0\right] = u_{(\gamma)}\left(xe^{\left(r+\frac{1}{2\gamma}|\lambda|^2\right)T}\right).$$

Let

$$\begin{aligned}\bar{U}^{(T,\gamma)}(x) :=& \mathbb{E}\left[\sup_{\pi\in\mathscr{A}_T^{\mathcal{G}}} \mathbb{E}\left[u_{(\gamma)}(X_T^{x,\pi})|\mathcal{G}_0\right]\right]\\ =& \mathbb{E}\left[u_{(\gamma)}\left(xe^{\left(r+\frac{1}{2\gamma}|\lambda|^2\right)T}\right)\right]\end{aligned}$$

be the expected optimal utility for the inside investor, and we are interested in evaluating the ratio of two optimized expected utilities $U^{(T,\gamma)}(x)$ and $\bar{U}^{(T,\gamma)}(x)$. We then see the following.

Proposition 5.3. *It holds that*

$$U^{(T,1)}(x) \sim \bar{U}^{(T,1)}(x) \quad \text{as } T\to\infty \tag{5.9}$$

and that, for $\gamma > 1$,

$$U^{(T,\gamma)}(x) \sim \bar{U}^{(T,\gamma)}(x)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{(\gamma-1)n}{2}} \quad \text{as } T\to\infty. \tag{5.10}$$

We may interpret that, for log-utility-investors, "cost of uncertainty of λ" becomes negligible as $T\to\infty$, while for power-utility-investors who are risk-averse than log-utility-investors, the "cost of uncertainty of λ" does not disappears even when $T\to\infty$.

Proof. We see that, from Theorem 4.1,

$$\begin{aligned}\frac{\bar{U}^{(T,\gamma)}(x)}{U^{(T,\gamma)}(x)} =& \frac{\log x + \left(r + \frac{1}{2}\mathbb{E}|\lambda|^2\right) T}{\log x + \left(r + \frac{1}{2T}\int_0^T \mathbb{E}|\hat{\lambda}_t|^2 dt\right) T} \\ =& \frac{\frac{1}{T}\log x + \left(r + \frac{1}{2}\mathbb{E}|\lambda|^2\right)}{\frac{1}{T}\log x + \left(r + \frac{1}{2T}\int_0^T \mathbb{E}|\hat{\lambda}_t|^2 dt\right)} \to 1 \quad \text{as } T \to \infty\end{aligned}$$

since $\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}|\hat{\lambda}_t|^2 dt \to \mathbb{E}|\lambda|^2$ as we see in Proof of Proposition 5.1. Hence, (5.9) follows. To obtain (5.10) with $\gamma > 1$, we write the optimal expected power-utility of the "insider" as

$$\bar{U}^{(T,\gamma)}(x) = u_{(\gamma)}\left(xe^{rT}\right) \int_{\mathbb{R}^n} e^{-\frac{\gamma-1}{2\gamma}T|z|^2} f_\nu(z)dz.$$

By Laplace's method, we see that

$$\lim_{T\to\infty} \left\{\frac{T(\gamma-1)}{2\pi\gamma}\right\}^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{\gamma-1}{2\gamma}T|z|^2} f_\nu(z)dz = f_\nu(0).$$

Hence, it follows that

$$\bar{U}^{(T,\gamma)}(x) \sim u_{(\gamma)}\left(xe^{rT}\right)\left\{\frac{2\pi}{T}\left(\frac{\gamma}{\gamma-1}\right)\right\}^{\frac{n}{2}} f_\nu(0) \quad \text{as } T \to \infty.$$

Combining it with Proposition 5.2, we complete the proof. □

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A Appendix: HJB Approach

In this appendix, we sketch a standard dynamic programming approach for solving Bayesian CRRA-utility maximization (2.15), which is different to the martingale method, employed in [9]. First, we reformulate (2.15), introducing a measure-change. Note that

$$\mathbb{E}u_{(\gamma)}(X_T^{x,\pi}) = \tilde{\mathbb{E}}Z_T u_{(\gamma)}(X_T^{x,\pi}) = \tilde{\mathbb{E}}\tilde{Z}_T u_{(\gamma)}(X_T^{x,\pi}),$$

where we write

$$\tilde{Z}_t := \tilde{\mathbb{E}}[Z_t|\mathcal{F}_t] = F(t, \tilde{W}_t - \tilde{W}_0)$$

and use (2.7) and (2.11) since $X^{x,\pi}$ is $\mathcal{F}_t$-adapted. Combining (2.1), (2.3) and (2.14), we see

$$dX_t^{x,\pi} = X_t^{x,\pi}\left\{rdt + \pi_t^{\top}\sigma(t,S_t)d\tilde{W}_t\right\}, \quad X_0^{x,\pi} = x.$$

So,

$$d(X_t^{x,\pi})^{1-\gamma} = (X_t^{x,\pi})^{1-\gamma}\left[(1-\gamma)\left\{r - \frac{\gamma}{2}\left|\sigma_t^{\top}\pi_t\right|^2\right\}dt + (1-\gamma)\pi_t^{\top}\sigma_t d\tilde{W}_t\right].$$

Hence, we have

$$(X_T^{x,\pi})^{1-\gamma} = x^{1-\gamma}e^{(1-\gamma)rT}M_T((1-\gamma)\sigma^{\top}\pi)\exp\left\{-\frac{\gamma(1-\gamma)}{2}\int_0^T\left|\sigma_t^{\top}\pi_t\right|^2 dt\right\},$$

where we define

$$M_t(\alpha) := \exp\left(\int_0^t \alpha_u^{\top} d\tilde{W}_u - \frac{1}{2}\int_0^t |\alpha_u|^2 du\right). \tag{A.1}$$

Let $\mathscr{U}_T^{(1)}$ be the totality of n-dimensional progressively measurable process $p := (p_t)_{t\in[0,T]}$ on the time-interval $[0,T]$ so that $\int_0^T |p_t|^2 dt < \infty$ a.s. and that $\mathbb{E}M_T((1-\gamma)\alpha) = 1$. For $\alpha \in \mathscr{U}_T^{(1)}$, we define the probability measure $\tilde{\mathbb{P}}_T^{(\alpha)}$ on $(\Omega, \mathcal{F}_T)$ by the formula

$$\left.\frac{d\tilde{\mathbb{P}}_T^{(\alpha)}}{d\tilde{\mathbb{P}}}\right|_{\mathcal{F}_t} := M_t((1-\gamma)\alpha), \quad t \in [0,T].$$

By Cameron-Martin-Maruyama-Girsanov's theorem, the process $(\tilde{W}_t^{(\alpha)})_{t\in[0,T]}$, defined by

$$\tilde{W}_t^{(\alpha)} := \tilde{W}_t - (1-\gamma)\int_0^t \alpha_u du,$$

is an n-dimensional $(\tilde{\mathbb{P}}_T^{(\alpha)}, \mathcal{F}_t)$-Brownian motion. Recall that, when $\alpha := \sigma^{\top}\pi \in \mathscr{U}_T^{(1)}$, we have

$$\begin{aligned}\log \mathbb{E}_T(X_T^{\pi})^{1-\gamma} &= (1-\gamma)(\log x + rT) \\ &\quad + \log\tilde{\mathbb{E}}_T^{(\alpha)}\exp\left\{\log F(T, \tilde{W}_T - \tilde{W}_0) - \frac{\gamma(1-\gamma)}{2}\int_0^T |\alpha_t|^2 dt\right\},\end{aligned} \tag{A.2}$$

where $\tilde{\mathbb{E}}_T^{(\alpha)}(\cdot)$ denotes expectation with respect to $\tilde{\mathbb{P}}_T^{(\alpha)}$.

We now consider, for $0 \le t \le T < \infty$,

$$\text{(A.3)} \quad \bar{V}^{(T)}(t,y) := \inf_{\alpha \in \mathscr{U}_{T-t}} \log \tilde{\mathbb{E}}_T^{(\alpha)} \exp\left\{ \log F(T, Y_{T-t}^{(y)} - \tilde{W}_0) + \frac{\gamma(\gamma-1)}{2} \int_0^{T-t} |\alpha_s|^2 ds \right\},$$

where we set a suitable $\mathscr{U}_{T-t}$, a subset of $\mathscr{U}_{T-t}^{(1)}$, and define the process $(Y_s^{(y)})_{s\in[0,T-t]}$ by

$$dY_s^{(y)} = (1-\gamma)\alpha_s ds + d\tilde{W}_s^{(\alpha)}, \quad Y_0^{(y)} = y.$$

The associated HJB equation is written down as

$$\text{(A.4)} \quad \begin{aligned} -\partial_t V &= \frac{1}{2}\left(\Delta V + |\nabla V|^2\right) + \inf_{\alpha \in \mathbf{R}^n} \left\{ (1-\gamma)\alpha^\top \nabla V + \frac{\gamma(\gamma-1)}{2}|\alpha|^2 \right\}, \\ V(T,y) &= \log F(T, y - \tilde{W}_0). \end{aligned}$$

Here, we see

$$(1-\gamma)\alpha^\top \nabla V + \frac{\gamma(\gamma-1)}{2}|\alpha|^2 = \frac{\gamma(\gamma-1)}{2}\left|\alpha - \frac{1}{\gamma}\nabla V\right|^2 - \frac{\gamma-1}{2\gamma}|\nabla V|^2.$$

So, the minimizer in (A.4) is given by

$$\bar{\alpha} := \frac{1}{\gamma}\nabla V$$

and (A.4) is rewritten as

$$\text{(A.5)} \quad \begin{aligned} -\partial_t V &= \frac{1}{2}\Delta V + \frac{1}{2\gamma}|\nabla V|^2, \\ V(T,y) &= \log F(T, y - \tilde{W}_0). \end{aligned}$$

Noting that $L := \mathrm{e}^{\frac{1}{\gamma}V}$ satisfies

$$-\partial_t L = \frac{1}{2}\Delta L, \quad L(T,y) = F(T, y - \tilde{W}_0)^{\frac{1}{\gamma}},$$

we deduce the expression for the solution to (A.5)

$$V(t,y) = \gamma \log \tilde{\mathbb{E}}\left[F(T, \tilde{W}_T - \tilde{W}_0)^{\frac{1}{\gamma}} \mid \tilde{W}_t = y \right].$$

From (A.2) and (A.3), we see that the relation

$$U^{(T,\gamma)}(x) = u_{(\gamma)}(x\mathrm{e}^{rT}) \exp\left\{ \bar{V}^{(T)}(0, \tilde{W}_0) \right\}$$

holds. So, we can deduce the expression

$$U^{(T,\gamma)}(x) = u_{(\gamma)}(x\mathrm{e}^{rT}) \left\{ \tilde{\mathbb{E}}\left[F(T, \tilde{W}_T - \tilde{W}_0)^{\frac{1}{\gamma}} \right] \right\}^{\gamma},$$

which is nothing but the representation (4.9). After demonstrating the so-called verification steps, we can establish all assertions in Theorem 4.2.